Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00012-0

J. Appl. Maths Mechs, Vol. 65, No. 1, pp. 119–124, 2001

0021-8928/01/$—see front matter

AN ASYMPTOTIC ANALYSIS OF THE STRESS–STRAIN STATE OF A STRIP REINFORCED WITH RIBS†

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(*Received* 15 *March* 1999)

An approximate solution of the problem of the stress–strain state of an anisotropic strip reinforced with two-dimensional ribs is constructed using the method of asymptotic expansion of generalized functions, the averaging method and the method of singular expansions.

One of the simplest theoretical versions of this problem is as follows: the ribs operate only under tension-compression and the panel (the matrix) operates under shear [1, 2]. This model, while it enables one to cover many characteristic features of the phenomenon, is nevertheless too rough from the point of view of the mechanics of a deformed solid. For extremely anisotropic media, the method based on the expansion with respect to geometrical-stiffness parameters [3–5] is extremely effective, but its accuracy is limited in the isotropic case. Methods of the theory of functions of a complex variable [6] lead to the need to solve infinite systems of coupled linear algebraic equations, where, as a rule, closed solutions can only be obtained for one stringer [7]. Exact solutions of the periodic problem in double trigonometric series are only possible for certain special boundary conditions [8, 9]. The averaging method [10, 11] enables an extremely simple algebraic solution to be obtained for one-dimensional ribs, but its extension to the two-dimensional case [12] is very artificial.

Below we use the averaging method with a previously proposed modification [13] to obtain an analytical solution, while, to take into account the width of the rib, an asymptotic method is used based on expansions employing generalized functions [14, 15].

1. FORMULATION OF THE PROBLEM

Consider the stress–strain state of a strip, reinforced with two-dimensional ribs. The bending stiffness of the ribs is ignored, which is justified by the smallness of this quantity compared with the tension-compression stiffnesses. When there are no mass forces the initial equations for the anisotropic material of the panel (the matrix), the principal directions of anisotropy of which coincide with the Cartesian coordinate axes, can be written in the form [4]

$$[B_{11}+B_0\Phi_0(y)]\frac{\partial^2 u}{\partial x^2}+B_{33}\frac{\partial^2 u}{\partial y^2}+(B_{33}+B_{12})\frac{\partial^2 v}{\partial x\partial y}=0$$

$$B_{22}\frac{\partial^2 v}{\partial x^2}+B_{33}\frac{\partial^2 v}{\partial y^2}+(B_{33}+B_{12})\frac{\partial^2 u}{\partial x\partial y}=0 \tag{1.1}$$

$$\Phi_0(y)=\sum_{k=-\infty}^{\infty}[H(y+kb-\varepsilon)+H(y-kb+\varepsilon)]$$

Here $H(\ldots)$ is the Heaviside function, B_{ij} $(i=1,2, j=1,2)$ are stiffness parameters, characterizing the panel (the matrix), B_0 is the tension-compression stiffness of the ribs, u and v are the displacements in the direction of the x and y axes respectively and b is the distance between the centre lines of neighbouring ribs.

We will take the boundary conditions in the form

†*Prikl. Mat. Mekh.* Vol. 65, No. 1, pp. 123–128, 2001.

$$x=0, H: \quad v=0, \quad [B_{11}+B_0\Phi_0(y)]\frac{\partial u}{\partial x}=P(y)\Phi_0(y) \tag{1.2}$$

Here H is the width of the strip and $P(_y)$ is the specified boundary load.

From the physical point of view, these boundary conditions correspond to the transfer of the load to the ribs for a strip reinforced against displacements in the direction of the y axis. Note that, in the scheme assumed, there are no singularities of the solution at the points of contact of the ribs and the casing [7].

2. THE CHANGE TO ONE-DIMENSIONAL RIBS

We will assume initially that the width of the ribs is 2ε. Assuming that the ribs are thin and taking ε as the small parameter, we expand the function $\Phi_0(y)$ in series in ε [14, 15]. Applying a bilateral Laplace transformation, expanding the transform in series in ε and carrying out an inverse transformation (which is justified within the framework of the theory of generalized functions [14, 15]), we obtain

$$\Phi_0(y)=2\varepsilon\Phi(y)+2\varepsilon\sum_{n=1,3,5,\dots}\varepsilon^n\Phi^{(n)}(y), \qquad \Phi(y)=\sum_{k=-\infty}^{\infty}\delta(y-kb)$$

Here $\delta(x)$ is the Dirac delta function.

We now represent the solution of initial boundary-value problem (1.1), (1.2) in the form

$$u=u_0+\varepsilon u_1+\varepsilon^2u_2+\dots, \qquad v=v_0+\varepsilon v_1+\varepsilon^2 v_2+\dots \tag{2.1}$$

and change to the recurrent sequence of boundary-value problems

$$\begin{aligned}
&L_1(u_0,v_0)\equiv[B_{11}+B_c\Phi(y)]\frac{\partial^2u_0}{\partial x^2}+B_{33}\frac{\partial^2u_0}{\partial y^2}+(B_{33}+B_{12})\frac{\partial^2v_0}{\partial x\partial y}=0\\
&L_2(u_0,v_0)\equiv B_{22}\frac{\partial^2v_0}{\partial x^2}+B_{33}\frac{\partial^2v_0}{\partial y^2}+(B_{33}+B_{12})\frac{\partial^2u_0}{\partial x\partial y}=0\\
&L_1(u_1,v_1)=-\frac{1}{2}B_c\frac{\partial^2u_0}{\partial x^2}\Phi'(y), \qquad L_2(u_1,v_1)=0,\dots\\
&L_1(u_n,v_n)=-B_c\sum_{i=0}^{n-1}\frac{\partial^2u_i}{\partial x^2}\Phi^{(n-i)}(y), \qquad L_2(u_n,v_n)=0\\
&x=0,H: \quad v_i=0,\dots \quad i=0,1,2,\dots
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&L_3(u_0)\equiv[B_{11}+B_c\Phi(y)]\frac{\partial u_0}{\partial x}=P(y)\Phi(y)\\
&L_3(u_1)=\frac{1}{2}\left[-B_c\frac{\partial u_0}{\partial x}+P(y)]\Phi'(y)\right]\\
&L_3(u_n)=-\frac{1}{2}B_c\sum_{i=0}^{n-1}\frac{\partial u_i}{\partial x}\Phi^{(n-i)}(y)+P(y)\Phi^{(n)}(y); \qquad B_c=2\varepsilon B_0
\end{aligned} \tag{2.3}$$

3. AVERAGED RELATIONS

We will use the two-scale method [13, 16, 17]. We will first introduce some estimates. We will assume that the strip is fairly wide ($H \gg 2b$), and an external load which varies smoothly from rib to rib. We will use as the small parameter ε_1 the ratio $\varepsilon_1 = b/H$. The ribs are assumed to be fairly stiff $B_cb/B_{11} \sim \varepsilon_1^{-1}$. In other words, the reduced stiffness of a rib considerably exceeds the tension-compression stiffness of the strip.

In dimensionless variables $\xi = x/H$; $\eta = y/H$ the first two equations of system (2.2) can be rewritten in the form

$$L_{11}(u_0,v_0) \equiv B_{11}\frac{\partial^2 u_0}{\partial \xi^2} + B_{33}\frac{\partial^2 u_0}{\partial \eta^2} + (B_{33}+B_{12})\frac{\partial^2 v_0}{\partial \xi \partial \eta} = 0$$

$$L_{12}(u_0,v_0) \equiv B_{22}\frac{\partial^2 v_0}{\partial \xi^2} + B_{33}\frac{\partial^2 v_0}{\partial \eta^2} + (B_{33}+B_{12})\frac{\partial^2 u_0}{\partial \xi \partial \eta} = 0$$

$$\varepsilon_1 k < y < \varepsilon_1(k+1) \tag{3.1}$$

$$\left\{u_0^+, v_0^+, \frac{\partial v_0^+}{\partial y}\right\} = \left\{u_0^-, v_0^-, \frac{\partial v_0^+}{\partial y}\right\}$$

$$B_{11}\left(\frac{\partial u_0^+}{\partial \eta} - \frac{\partial u_0^-}{\partial \eta}\right) = \frac{B_c}{H}\frac{\partial^2 u_0}{\partial \xi^2} \qquad \left((\ldots)^\pm = \lim_{\eta \to \varepsilon_1 k \pm 0}(\ldots)\right)$$

In accordance with the two-scale method we introduce a "fast" variable $\eta_1 = y/b$, retaining the notation η for the "slow" variable. Then

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} + \varepsilon_1^{-1}\frac{\partial}{\partial \eta_1} \tag{3.2}$$

We will seek a solution in the form

$$u_0 = u^{(0)} + \varepsilon_1^2 u^{(1)} + \varepsilon_1^3 u^{(2)} + \ldots, \qquad v_0 = v^{(0)} + \varepsilon_1^2 v^{(1)} + \varepsilon_1^3 v^{(2)} + \ldots \tag{3.3}$$

Here

$$u^{(0)} \equiv u^{(0)}(\xi,\eta), \qquad v^{(0)} \equiv v^{(0)}(\xi,\eta)$$

$$u^{(i)} \equiv u^{(i)}(\xi,\eta,\eta_1), \qquad v^{(i)} \equiv v^{(i)}(\xi,\eta,\eta_1), \qquad i = 1, 2, \ldots \tag{3.4}$$

$$u^i(\xi,\eta,\eta_1+1) = u^{(i)}(\xi,\eta,\eta_1), \qquad v^{(i)}(\xi,\eta,\eta_1+1) = v^{(i)}(\xi,\eta,\eta_1)$$

Using relations (3.1)–(3.4), after splitting with respect to ε_1 we obtain

$$\frac{\partial^2 u^{(1)}}{\partial \eta_1^2} = -L_{11}(u^{(0)},v^{(0)}), \qquad \frac{\partial^2 v^{(1)}}{\partial \eta_1^2} = -L_{12}(u^{(0)},v^{(0)}) \tag{3.5}$$

Equations (3.5) can be integrated without difficulty

$$u^{(1)} = -\frac{1}{2}\eta_1^2 L_{11}(u^{(0)},v^{(0)}) + C_1(\xi,\eta)\eta_1 + C_2(\xi,\eta) \tag{3.6}$$

$$v^{(1)} = -\frac{1}{2}\eta_1^2 L_{12}(u^{(0)},v^{(0)}) + C_3(\xi,\eta)\eta_1 + C_4(\xi,\eta)$$

The third and fourth conditions of (3.1) can be rewritten in the form

$$\left\{u^{(1)}, v^{(1)}, \frac{\partial v^{(1)}}{\partial \eta_1}\right\}\Bigg|_{\eta_1=0} = \left\{u^{(1)}, v^{(1)}, \frac{\partial v^{(1)}}{\partial \eta_1}\right\}\Bigg|_{\eta_1=0} \tag{3.7}$$

$$B_{11}\left(\frac{\partial u^{(1)}}{\partial \eta_1}\Bigg|_{\eta_1=1} - \frac{\partial u^{(1)}}{\partial \eta_1}\Bigg|_{\eta_1=0}\right) = B_1\frac{\partial^2 u^{(0)}}{\partial \xi^2}, \qquad B_1 = \frac{B_c}{b}$$

The constants C_1 and C_2 are not determined from conditions (3.7) (they must be related to the following averaged approximation [13]). From the first condition of (3.7) we obtain the equation

$$L_{12}(u^{(0)}, v^{(0)}) = 0 \tag{3.8}$$

and also the quantities

$$C_3 = 0, \quad C_1 = L_{11}(u^{(0)}, v^{(0)})$$

By satisfying the second condition of (3.7) we obtain the averaged equation

$$L_{11}(u^{(0)}, v^{(0)}) + B_1 \frac{\partial^2 u^{(0)}}{\partial \xi^2} = 0 \tag{3.9}$$

Equations (3.8) and (3.9) form the required averaged system of equations.
The first "fast" correction can be written in the form

$$u_1 = \eta_1\left(\frac{1}{2} - \eta_1\right) B_1 \frac{\partial^2 u^{(0)}}{\partial \xi^2}, \quad v_1 = 0 \tag{3.10}$$

We will now analyse boundary conditions (2.3). Form the first boundary condition of (2.3) we obtain

$$\xi = 0, 1: \quad v^{(0)} = 0 \tag{3.11}$$

Averaging the second of relations (2.3), we have

$$\xi = 0, 1 \quad (B_{11} + B_1)\partial u^{(0)} / \partial \xi = P_1(\eta), \quad P_1(\eta) = P_1 H / b \tag{3.12}$$

Equations (3.9) and (3.10), with boundary conditions (3.11) and (3.12), give the required averaged boundary-value problem.

4. THE BOUNDARY LAYER

The second boundary condition of (2.3) is therefore satisfied on average. The discrepancies in satisfying the boundary conditions in this case are obtained to be self-balancing in the sections $k - 1 < \eta < k$ ($k = 0, \pm 1, \ldots$). Consequently, the stress–strain state corresponding to these fictitious loads will be concentrated in the region of the strip edges and will attenuate with respect to ξ at distances of the order of ε_1. In other words, we are dealing with a stressed state of the boundary-layer type, which we will now analyse.

We introduce a new "fast" variable $\xi_1 = \xi/\varepsilon_1$, retaining the notation ξ for the "slow" variable. Then

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} + \varepsilon_1^{-1} \frac{\partial}{\partial \xi_1} \tag{4.1}$$

We will seek a solution of the boundary-layer type in the form

$$u_n = \varepsilon_1 u_n^{(0)} + \varepsilon_1^2 u_n^{(2)} + \ldots, \quad v_n = \varepsilon_1 v_n^{(0)} + \varepsilon_1^2 v_n^{(2)} + \ldots \tag{4.2}$$

The functions on the right-hand sides of (4.2) depend on the variables ξ, ξ_1, η, η_1.

Substituting expression (4.1) and expansions (4.2) into the initial relations, we obtain, after splitting with respect to ε_1, in the first approximation

$$L_{11}^{(1)}(u_n^{(0)}, v_n^{(0)}) = 0, \quad L_{12}^{1}(u_n^{(0)}, v_n^{(0)}) = 0$$

$$u_n^{(0)} = v_n^{(0)} = 0 \text{ when } \eta_1 = k, \quad k = 0, \pm 1, \ldots \tag{4.3}$$

$$v_n^{(0)} = 0, \quad B_{11} \frac{\partial u_n^{(0)}}{\partial \xi_1} = -\varepsilon_1 P_1(\eta) \text{ when } \xi_1 = 0, \ \varepsilon_1^{-1} \tag{4.4}$$

$$u_n^{(0)},\ v_n^{(0)} \to 0 \quad \text{as} \quad \xi_1 \to \infty \tag{4.5}$$

$$u_n^{(0)},\ v_n^{(0)} \to 0 \quad \text{as} \quad \xi_1 \to -\infty \tag{4.6}$$

The operators $L_{11}^{(1)}, L_{12}^{(1)}$ differ from the operators L_{11} and L_{12} in (3.1) by the replacement of ξ and η by ξ_1 and η_1.

We can solve the problem during one period (say, for $k = 0, 1$) and confine ourselves to boundary conditions (4.3)–(4.5) (the solution is then written automatically for boundary conditions (4.3), (4.4) and (4.6)).

Thus, to calculate the boundary layer we arrive at the classical problem of elasticity theory of a half-strip, clamped on the long side, the solution of which can be obtained by different methods.

In the case considered, when an approximate solution is constructed it is quite appropriate to confine ourselves to splitting with respect to the dimensionless parameter [3, 4]. In this approach we solve the following equations in the fundamental approximation

$$B_{11}\frac{\partial^2 u_n^{(0)}}{\partial \xi_1^2} + B_{33}\frac{\partial^2 u_n^{(0)}}{\partial \eta_1^2} = 0 \tag{4.7}$$

with boundary condition

$$\eta_1 = 0, 1: \quad u_n^{(0)} = 0; \quad \xi_1 = 0: \quad B_{11}\frac{\partial^2 u_n^{(0)}}{\partial \xi_1} = -\varepsilon_1 P_1(\eta) \tag{4.8}$$

$$u_n^{(0)} \to 0 \quad \text{as} \quad \xi_1 \to \infty$$

As a result we have

$$u_n^{(0)} = \bar{p} \sum_{k=1,3,5,\ldots} \frac{e^{-\lambda_k \xi}\sin(k\pi x)}{k^2}; \quad \bar{p} = \frac{2\varepsilon P_1(\eta)}{\pi^2\sqrt{B_{11}B_{33}}}; \quad \lambda_k = \sqrt{\frac{B_{33}}{B_{11}}}\,\pi k \tag{4.9}$$

Hence, the use of asymptotic methods enables us to obtain an approximate analytical solution of the problem in question.

5. EXAMPLE

As an example consider the deformation of an isotropic half-strip with the following boundary conditions

$$x = 0: \quad T_1 = E\cos\eta, \quad S = 0$$

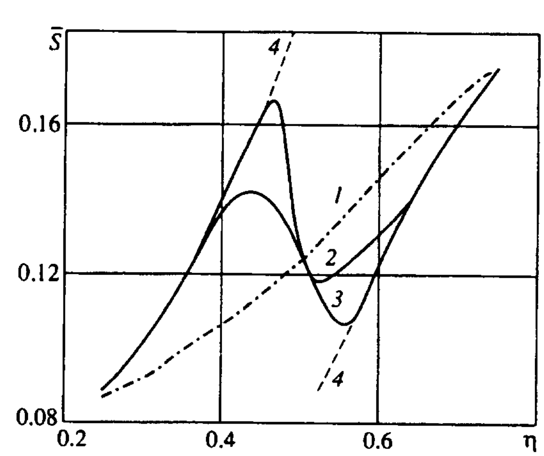

Fig. 1

where $\eta = y/l$, l is a quantity characterizing the variability of the external load, T_1 is the tension-compression force in the direction of the x axis and S is the shear force.

Figure 1 shows the dimensionless quantities $\bar{S} = S/E$. Note that it is this force which largely depends on the discreteness of the arrangement of the ribs. We used the following values of the parameters: $B_c = B$, $\nu = 0.3$, $b/l = 0.5$, $h/l = 1/32$ and $F/(lh) = 3/32$. Curve 1 corresponds to the constructive-orthotropic solution, and curves 2 and 3 correspond to the solutions which take into account the discreteness of the arrangement of the ribs when $\varepsilon/l = 3/32$ and $\varepsilon/l = 3/16$, respectively. For comparison we also give graphs of the change in the dimensionless shear force, calculated for the scheme of contact along a line (curve 4).

It can be seen that, in the neighbourhood of the reinforcement it is necessary to take into account the discreteness of the arrangement of the ribs and their widths.

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Translated by R.C.G.